Principal differential calculi over projective bases

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Plan of the talk

(H, Δ, ϵ, S) Hopf algebra
A right H-comodule algebra

Main question: Given a Hopf-Galois extension $B := A^{\mathrm{co}H} \subseteq A$ can we find noncommutative differential calculi $\Omega^\bullet(A)$, $\Omega^\bullet(H)$ such that

$$\Omega^\bullet(B) = \Omega^\bullet(A)^{\mathrm{co}\Omega^\bullet(H)} \subseteq \Omega^\bullet(A)$$

is a Hopf-Galois extension of graded algebras?

We give conditions for this to holds as first order differential calculi: Principal differential calculi

- Principal comodule algebras (faithfully flatness)
- Base forms, horizontal forms and compatibility

Then we discuss sheaves of calculi which are locally principal.

- local $\leftrightarrow$ global principles in noncommutative geometry
- use Ore localization of algebras/differential calculi

Hopf-Galois Extensions

Let $\Bbbk$ be a field.

(A, δ_A) right H-comodule algebra with coaction $\delta_A \colon A \to A \otimes H$.
We write $\Delta(h) = h_1 \otimes h_2$ and $\delta_A(a) = a_0 \otimes a_1$.

$B := A^{\mathrm{co}H} := \{a \in A \mid \delta_A(a) = a \otimes 1\}$

Definition (Kreimer-Takeuchi '80)

$B \subseteq A$ is called a Hopf-Galois extension if the canonical map

$$\chi \colon A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto aa'_0 \otimes a'_1$$

is a bijection.

Example

i.) If $A = H$ then $\Bbbk = A^{\mathrm{co}H} \subseteq A$ is Hopf-Galois extension with $\chi^{-1}(a \otimes h) = aS(h_1) \otimes h_2$.

ii.) $\pi \colon P \to M$ principal G-bundle, $A = \mathcal{C}^\infty(M)$, $H = \mathcal{C}^\infty(G)$.
Right G-action $r \colon P \times G \to P$ induces right coaction $\delta_A := r^* \colon A \to A \otimes H$.
$B := A^{\mathrm{co}H} = \mathcal{C}^\infty(P/G) = \mathcal{C}^\infty(M)$
$\phi \colon P \times G \to P \times_M P$, $(p, g) \mapsto (p, r(p, g))$ induces $\chi := \phi^*$ and ϕ is bijection if r is free and transitive.

Principal Comodule Algebras

Definition

(A, δ_A) is called a principal comodule algebra if

- i.) $B := A^{\mathrm{co}H} \subseteq A$ is a Hopf-Galois extension and
- ii.) A is a faithfully flat left B-module, i.e. $\mathcal{M}_B \to \mathcal{M}_A^H$, $M \mapsto M \otimes_B A$ is an exact functor which reflects exactness.

In case the antipode of H is invertible we have equivalently to ii.)

- ii'.) There is a section $s \colon A \to B \otimes A$ of the left action $\ell \colon B \otimes A \to A$ in ${}_B\mathcal{M}^H$, i.e. $\ell \circ s = \mathrm{id}_A$.
- ii".) There is a right H-colinear unitary map $j \colon H \to A$.

Theorem (Schneider '90)

The following are equivalent.

- i.) (A, δ_A) *is a principal comodule algebra.*
- ii.) $\mathcal{M}_B \cong \mathcal{M}_A^H$ *are equivalent categories.*

Examples of Principal Comodule Algebras

1. **The smash product algebra:** Let B be a left H-module algebra. Then $A = B\#H$ is a right H-comodule algebra with $B = A^{\mathrm{co}H}$
$$(b\#h)(b'\#h') = b(h_1 \triangleright b')\#h_2 h'$$
It is a principal comodule algebra with cleaving map $j \colon H \to A$, $h \mapsto 1\#h$.
2. $A = \mathrm{SL}_q(2)$ free algebra generated by $\alpha, \beta, \gamma, \delta$ modulo
$$\alpha\beta = q^{-1}\beta\alpha, \quad \alpha\gamma = q^{-1}\gamma\alpha, \quad \beta\delta = q^{-1}\delta\beta, \quad \gamma\delta = q^{-1}\delta\gamma,$$
$$\beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma, \quad \alpha\delta - q^{-1}\beta\gamma = 1$$
is Hopf algebra with $\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.
Consider the Hopf algebra quotient $\pi \colon A \to H = U(1)$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$
Then A is a right H-comodule algebra with $\delta_A = (\mathrm{id} \otimes \pi) \circ \Delta \colon A \to A \otimes H$.
and $B = A^{\mathrm{co}H} = \mathcal{O}_q(\mathbb{S}^2)$ is the Podleś sphere.
One can show that A is faithfully flat as a B-module.

First Order Differential Calculi

A associative unital algebra.

Definition

We call (Γ, d) a first order differential calculus (FODC) on A, if

1. Γ is A-bimodule;
2. $\mathrm{d}\colon A \to \Gamma$ is $\Bbbk$-linear s.t.
$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}b \qquad \text{(Leibniz rule)}$$
holds for all $a, b \in A$;
3. $\Gamma = A\mathrm{d}A := \mathrm{span}_\Bbbk\{a\mathrm{d}b \mid a, b \in A\}$; (Surjectivity)

Example

i.) $A = \mathscr{C}^\infty(M)$, $\Gamma = \Gamma^\infty(T^*M)$, $\mathrm{d}\colon A \to \Gamma$ de Rham differential $\mathrm{d}f|_U = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$ in local chart (U, x).

ii.) $A = \mathbb{C}_q\mathbb{S}^1 := \mathbb{C}[t, t^{-1}]$, $q \in \mathbb{C}^\times$ not root of unity $\Gamma = A\mathrm{d}t$, $\mathrm{d}t \cdot f(t) := f(qt)\mathrm{d}t$, and $\mathrm{d}f|_t := \frac{f(qt)-f(t)}{t(q-1)}\mathrm{d}t$, for f rational function in t.

Covariant Differential Calculi

H Hopf algebra

(A, δ_A) right H-comodule algebra

Definition (Woronowicz '89)

A FODC (Γ, d) on A is right H-covariant if

$$a\mathrm{d}a' \mapsto a_0\mathrm{d}a'_0 \otimes a_1 a'_1 \tag{1}$$

for $a, a' \in A$ extends to a well-defined $\mathbb{k}$-linear map $\Gamma \to \Gamma \otimes H$.

Proposition

A FODC (Γ, d) on (A, δ_A) is right H-covariant if and only if

- (Γ, Δ_Γ) *is a right H-covariant A-bimodule:* $\Delta_\Gamma(a \cdot \omega \cdot a') = \delta_A(a) \cdot \Delta_\Gamma(\omega) \cdot \delta_A(a')$
- d *is right H-colinear:* $\Delta_\Gamma \circ \mathrm{d} = (\mathrm{d} \otimes \mathrm{id}_H) \circ \delta_A$

Then Δ_Γ is determined by (1).

Lemma

Let (Γ, d) be a right H-covariant FODC on a right H-comodule algebra A.

i.) *An injective right H-comodule algebra map $\iota \colon A' \hookrightarrow A$ induces a right H-covariant FODC $(\Gamma_\iota, \mathrm{d}_\iota)$ on A', where*

$$\Gamma_\iota := \iota(A')\mathrm{d}\iota(A') \subseteq \Gamma$$

and $\mathrm{d}_\iota \colon A' \ni a' \mapsto \mathrm{d}\iota(a') \in \Gamma_\iota$.

ii.) *A surjective right H-comodule algebra map $\pi \colon A \to A'$ induces a right H-covariant FODC $(\Gamma_\pi, \mathrm{d}_\pi)$ on A', where*

$$\Gamma_\pi := \Gamma / \Gamma_I$$

is the A-bimodule quotient with $\Gamma_I := I\mathrm{d}A + A\mathrm{d}I$, where $I := \ker \pi \subseteq A$ and $\mathrm{d}_\pi \colon A' \ni \pi(a) \mapsto [\mathrm{d}a] \in \Gamma_\pi$.

iii.) *If ι is a section of π we have an isomorphism $(\Gamma_\iota, \mathrm{d}_\iota) \cong (\Gamma_\pi, \mathrm{d}_\pi)$ of right H-covariant FODC.*

We call $(\Gamma_\iota, \mathrm{d}_\iota)$ the pullback calculus, while we call $(\Gamma_\pi, \mathrm{d}_\pi)$ the quotient calculus.

Horizontal and Vertical Forms

(A, δ_A) principal comodule algebra,
recall this means $B = A^{\mathrm{co}H} \subseteq A$ Hopf-Galois and A is faithfully flat B-module.

Definition

A FODC (Γ_A, d_A) on A and a left covariant FODC (Γ_H, d_H) on H are called a weak principal DC if ver is well-defined and makes

$$0 \to A \otimes_B \Gamma_B \to \Gamma_A \xrightarrow{\mathrm{ver}} A \square_H \Gamma_H \to 0$$

exact. They are called principal DC if in addition (Γ_A, d_A) is right H-covariant and (Γ_H, d_H) is bicovariant.

Above

$$\mathrm{ver} \colon \Gamma_A \to A \square_H \Gamma_H, \qquad a \mathrm{d}_A a' \mapsto a_0 a'_0 \otimes a_1 \mathrm{d}_H a'_1,$$

where $A \square_H \Gamma_H := \mathrm{span}_{\Bbbk}\{a \otimes \omega_H \in A \otimes \Gamma_H \mid \delta_A(a) \otimes \omega_H = a \otimes \Delta_L^{\Gamma_H}(\omega_H)\}$.

Warning: ver might not be well-defined!

Principal Differential Calculi

Lemma

Let $\pi\colon A\to H$ *be a Hopf algebra quotient and* (Γ_A,d_A) *a left covariant FODC on* A. *Then*

i.) *With* $\Delta_L:=(\pi\otimes\mathrm{id})\circ\Delta_L^{\Gamma_A}\colon\Gamma_A\to H\otimes\Gamma_A$ (Γ_A,d_A) *becomes left* H*-covariant.*

ii.) *The quotient calculus* (Γ_H,d_H) *on* H *is left covariant.*

iii.) ver *is well-defined and* $\mathrm{ver}=(\mathrm{id}\otimes\pi_\Gamma)\circ\Delta_L^{\Gamma_A}\colon\Gamma_A\to A\Box_H\Gamma_H$.

Definition

We have

i.) the pullback calculus (Γ_B,d_B) w.r.t. $\iota\colon B=A^{\mathrm{co}H}\to A$ (Base forms)

ii.) the (A,B)-sub-bimodule
$\Gamma^{\mathrm{hor}}:=A\Gamma_B:=\mathrm{span}_{\Bbbk}\{a\omega_B\mid a\in A,\omega_B\in\Gamma_B\}\subseteq\Gamma_A$ (Horizontal forms)

Proposition

Exactness of $0\to A\otimes_B\Gamma_B\to\Gamma_A\xrightarrow{\mathrm{ver}}A\Box_H\Gamma_H\to 0$ *is equivalent to the exactness of* $0\to A\Gamma_B\to\Gamma_A\xrightarrow{\mathrm{ver}}A\otimes{}^{\mathrm{co}H}\Gamma_H\to 0$ *(= strong quantum principal bundle à la Majid). Then* $A\otimes_B\Gamma_B\cong A\Gamma_B$.

Examples

Example

i.) Consider the principal comodule algebra $\mathcal{O}_q(\mathbb{S}^2) \subseteq \mathrm{SL}_q(2)$ with structure Hopf algebra $U(1)$.

- The 3-dimensional right covariant FODC on A is a principal DC.
- The 4-dimensional bicovariant FODC on A is not a weak principal DC.

ii.) Let B be a left H-module algebra. The smash product $A = B\#H$ is a right H-comodule algebra with $A^{\mathrm{co}H} = B$.

$$(b\#h)(b'\#h') = (b(h_1 \triangleright b')\#h_2 h')$$

Choose a bicovariant FODC (Γ_H, d_H) on H and an H-module FODC (Γ_B, d_B) on B, i.e. $(h \triangleright \mathrm{d}_B b = \mathrm{d}_B(h \triangleright b))$.
There is a natural right H-covariant FODC $(\Gamma_\#, \mathrm{d}_\#)$ on A, given by

$$\Gamma_\# = \Gamma_B\#H \oplus B\#\Gamma_H \text{ and } \mathrm{d}_\# = \mathrm{d}_B \oplus \mathrm{d}_H.$$

This is a principal DC on A.

Graded Hopf-Galois Extension

(A, δ_A) principal comodule algebra. (Γ_A, d_A) and (Γ_H, d_H) principal DC.

Lemma

i.) *$\Omega_H^{\leq 1} = H \oplus \Gamma_H$ is a graded Hopf algebra with*

$$\Delta^1 = \Delta_R^{\Gamma_H} + \Delta_L^{\Gamma_H} \colon \Gamma_H \to \Gamma_H \otimes H \oplus H \otimes \Gamma_H$$

and $S^1 \colon \Gamma_H \to \Gamma_H$, $\omega \mapsto -S(\omega_{-1})\omega_0 S(\omega_1)$.

ii.) *$\Omega_A^{\leq 1} = A \oplus \Gamma_A$ is a graded right $\Omega_H^{\leq 1}$-comodule algebra with*

$$\delta_A^1 = \Delta_R^{\Gamma_A} + \mathrm{ver} \colon \Gamma_A \to \Gamma_A \otimes H \oplus A \otimes \Gamma_H.$$

Theorem (Aschieri-Fioresi-Latini-W)

For a principal DC: $\Omega_B^{\leq 1} = (\Omega_A^{\leq 1})^{\Omega_H^{\leq 1}} \subseteq \Omega_A^{\leq 1}$ is a graded Hopf-Galois extension.

We use tools developed by Schauenburg '96.

Corollary

For a principal DC we have $\Gamma_B = \Gamma_A^{\mathrm{co}H} \cap \Gamma_A^{\mathrm{hor}}$.

Quantum Principal Bundles

$\mathrm{pr}\colon E \to M$ surjective morphisms of algebraic varieties, P affine group with associated Hopf algebra H.

Theorem (Pflaum '94)

pr *is* P*-principal bundle if and only if* $\mathcal{F}(U) := \mathcal{O}_E(\mathrm{pr}^{-1}(U))$ *defines a sheaf of right* H*-comodule algebras such that on an open cover* $\{U_i\}$ *of* M

1. $\mathcal{F}(U_i)^{\mathrm{co}H} \cong \mathcal{O}_M(U_i)$
2. $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{co}H} \otimes H$

Condition 2. says that $\mathcal{F}(U_i)^{\mathrm{co}H} \subseteq \mathcal{F}(U_i)$ is a cleft Hopf-Galois extension, i.e. there is a convolution invertible right H-colinear map $j\colon H \to \mathcal{F}(U_i)$.

$(M, \mathcal{O}_M)$ quantum ringed space, H Hopf algebra.

Definition (Aschieri-Fioresi-Latini '21)

Sheaf $\mathcal{F}$ of right H-comodule algebras is (locally cleft) quantum principal bundle over M if there is open cover $\{U_i\}$ of M such that 1. and 2. hold.

If j is algebra map (locally trivial QPB) then $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{co}H} \# H$ as comodule algebras, where $h \triangleright b := j(h_1) b j^{-1}(h_2)$.

Example $\mathrm{SL}_q(2)$ over $\mathbb{CP}^1$

Consider $A := \mathrm{SL}_q(2)$ with Hopf algebra quotient $\pi\colon A \to H$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix}$, where $H := \mathcal{O}_q(P) := \mathbb{C}_q[t, t^{-1}, p]/(tp - q^{-1}pt)$ on parabolic subgroup P.

Consider the topology $\{\emptyset, U_1, U_2, U_{12}, \mathbb{CP}^1\}$ on $\mathbb{CP}^1$.
We define the sheaves

$$\mathcal{F}(\emptyset) := \{0\},\ \mathcal{F}(U_1) := A[\alpha^{-1}],\ \mathcal{F}(U_2) := A[\gamma^{-1}],$$
$$\mathcal{F}(U_{12}) := (A[\alpha^{-1}])[\gamma^{-1}],\ \mathcal{F}(\mathbb{CP}^1) := A$$

of right H-comodule algebras and

$$\mathcal{O}_{\mathbb{CP}^1}(\emptyset) := \{0\},\ \mathcal{O}_{\mathbb{CP}^1}(U_1) := \mathbb{C}_q[\alpha^{-1}\gamma] = \mathbb{C}_q[u],$$
$$\mathcal{O}_{\mathbb{CP}^1}(U_2) := A[\gamma^{-1}\alpha] = \mathbb{C}_q[v],$$
$$\mathcal{O}_{\mathbb{CP}^1}(U_{12}) := \mathbb{C}_q[u, u^{-1}],\ \mathcal{O}_{\mathbb{CP}^1}(\mathbb{CP}^1) := \mathbb{C}_q$$

of algebras with restriction morphism $r_{12,2}\colon v \mapsto u^{-1}$.

$\Rightarrow \mathcal{F}$ is QPB over $\mathcal{O}_{\mathbb{CP}^1}$ with cleaving maps $j_1\colon t^{\pm} \mapsto \alpha^{\pm}$, $p \mapsto \beta$ and $j_2\colon t^{\pm} \mapsto \gamma^{\pm}$, $p \mapsto \delta$.

General Construction

G complex semisimple algebraic group, P parabolic subgroup.
$\Rightarrow G/P$ is projective variety and $G \to G/P$ principal bundle.
Let $\mathcal{O}_q(G)$, $\mathcal{O}_q(P)$ be Hopf algebra quantizations of $\mathcal{O}(G)$, $\mathcal{O}(P)$.

$s \in \mathcal{O}_q(G)$ is quantum section if $(\mathrm{id} \otimes \pi)\Delta(s) = s \otimes \pi(s)$ and $s = t \mod (q-1)$ for a classical section t (see Ciccoli-Fioresi-Gavarini '08). We write $\Delta(s) = s^i \otimes s_i$.
$\Rightarrow \{s_i\}$ determine an algebra $\mathcal{O}_q(G/P)$ and an open cover $\{U_i\}$ of $M = G/P$.

Define $U_I := U_{i_1} \cap \ldots \cap U_{i_r}$ for $I = (i_1, \ldots, i_r)$.

Theorem (Aschieri-Fioresi-Latini '21)

1. *$U_I \mapsto \mathcal{O}_M(U_I) := \mathbb{C}_q[s_{k_1} s_{i_1}^{-1}, \ldots, s_{k_r} s_{i_r}^{-1}]$ for $1 \leq k_j \leq n$ defines a sheaf $\mathcal{O}_M$ of algebras on $M = G/P$.*
2. *$U_I \mapsto \mathcal{F}_G(U_I) := \mathcal{O}_q(G)\{s_I^r \mid r \leq 0\}$ defines a sheaf $\mathcal{F}_G$ of right H-comodule algebras.*
3. *$\mathcal{F}_G(U_I)^{\mathrm{co}\mathcal{O}_q(P)} = \mathcal{O}_M(U_I)$, i.e. $\mathcal{F}_G$ is a QPB over M, possibly non-cleft.*

Ore Extension of Calculi

Let (A, δ_A) be a right H-comodule algebra and $a \in A$ be an Ore element such that $\delta_A(a) \in A \otimes H$ is invertible.

Then $A[a^{-1}]$ is a right H-comodule algebra with $\delta_{A[a^{-1}]}(a^{-1}) = \delta_A(a)^{-1}$.

Lemma

Consider a right H-covariant FODC (Γ, d) on A and let $a \in A$ be as before. We define the $A[a^{-1}]$-bimodule

$$\Gamma_a := A[a^{-1}]\,\Gamma\,A[a^{-1}] := A[a^{-1}] \otimes_A \Gamma \otimes_A A[a^{-1}]$$

and the $\mathbb{k}$-linear map

$$\mathrm{d}_a : A[a^{-1}] \to \Gamma_a, \qquad \mathrm{d}_a(f) = \begin{cases} \mathrm{d}f & f \in A \\ -a^{-1}\,\mathrm{d}a\,a^{-1} & f = a^{-1} \end{cases},$$

where we extend d_a to $A[a^{-1}]$ by the Leibniz rule.

Then (Γ_a, d_a) is a right H-covariant FODC on $A[a^{-1}]$.

Calculi on Sheaves of Comodule Algebras

Stalk of a sheaf: for $x \in M$

$$\mathcal{F}_x = \{(U, s) \mid x \in U \text{ open and } s \in \mathcal{F}(U)\} / \sim$$

where $(U, s) \sim (V, t)$ iff $\exists W \subseteq U \cap V$ s.t. $s|_W = t|_W$.

Definition

A right H-covariant FODC on sheaf $\mathcal{F}$ of right H-comodule algebras is a sheaf Υ of $\mathcal{F}$-bimodules together with a morphism $\mathrm{d}\colon \mathcal{F} \to \Upsilon$ of sheaves of right H-comodules, such that on the stalks

1. $\mathrm{d}_x(fg) = \mathrm{d}_x(f)g + f\mathrm{d}_x g$ for all $f, g \in \mathcal{F}_x$
2. $\Upsilon_x = \mathcal{F}_x \mathrm{d}_x \mathcal{F}_x$

hold for all $x \in M$, where $\mathrm{d}_x\colon \mathcal{F}_x \to \Upsilon_x$ is the induced map on the stalks.

Theorem (Aschieri-Fioresi-Latini-W '21)

Let (Γ, d) be a right $\mathcal{O}_q(P)$-covariant FODC on the Hopf algebra $\mathcal{O}_q(G)$ and $\mathcal{F}_G$ as before. Then

1. *there is a right $\mathcal{O}_q(P)$-covariant FODC $(\Upsilon_G, \mathrm{d}_G)$ on the sheaf $\mathcal{F}_G$, where $(\Upsilon_G(U_I), \mathrm{d}_I)$ are the localizations of (Γ, d).*
2. *$(\Upsilon_G, \mathrm{d}_G)$ induces a FODC $(\Upsilon_M, \mathrm{d}_M)$ on the sheaf $\mathcal{O}_M$ and a right covariant FODC (Γ_H, d_H) on the Hopf algebra $\mathcal{O}_q(P)$.*

Principal Differential Calculi on Sheaves

Definition

Let $\mathcal{F}$ be a sheaf of principal comodule algebras. We say that a FODC (Υ, d) on $\mathcal{F}$ and a left covariant FODC (Γ_H, d_H) on H form a weak principal DC on $\mathcal{F}$, if on the topological basis $\{U_I\}$ of M

$$0 \to \mathcal{F}(U_I) \otimes_{\mathcal{O}_M(U_I)} \Upsilon_M(U_I) \to \Upsilon(U_I) \xrightarrow{\mathrm{ver}_I} \mathcal{F}(U_I) \Box_H \Gamma_H \to 0$$

is exact. If in addition $\Upsilon(U_I)$ is right H-covariant and (Γ_H, d_H) is bicovariant we have a principal DC on $\mathcal{F}$.

Theorem (Aschieri-Fioresi-Latini-W)

For a principal DC (Υ, d) on a sheaf $\mathcal{F}$ of principal comodule algebras it follows that

$$\Upsilon_M = \Upsilon^{\mathrm{co}H} \cap \Upsilon^{\mathrm{hor}}$$

is an equation of sheaves of $\mathcal{O}_M$-bimodules.

The 3-dim Covariant Calculus on $A = \mathrm{SL}_q(2)$

Theorem

Let (Γ_A, d_A) be the 3-dimensional left covariant FODC on A, consider the quotient calculus (Γ_H, d_H) on H and the left H-covariant FODC $(\Upsilon_G, \mathrm{d}_G)$ on $\mathcal{F}_G$. Then

i.) $\Upsilon_G(U_I) = \Gamma_{A_I}$ *is a free left* $\mathcal{F}_G(U_I) = A_I$*-module generated by* $\{\omega^0, \omega^1, \omega^2\}$.

ii.) *The base forms* $(\Upsilon_M, \mathrm{d}_M)$ *are determined by* $\Gamma_{B_1} = \mathrm{span}_{B_1}\{\alpha^{-2}\omega^2\}$ *and* $\Gamma_{B_2} = \mathrm{span}_{B_2}\{\gamma^{-2}\omega^2\}$ *as free left modules with commutation relations*

$$(\mathrm{d}_1 u)u = q^2 u \mathrm{d}_1 u, \qquad (\mathrm{d}_2 v)v = q^{-2} v \mathrm{d}_2 v,$$

where $u = \gamma\alpha^{-1} \in B_1$ *and* $v = \alpha\gamma^{-1} \in B_2$. *Furthermore* $\Gamma_{B_{12}} = \mathrm{span}_{B_{12}}\{\alpha^{-2}\omega^2\}$ *is a free left* B_{12}*-module and* $\mathrm{d}_1 u = -q^2 u^2 \mathrm{d}_2 v$ *in* $\Gamma_{B_{12}}$.

iii.) $(\Upsilon_G, \mathrm{d}_G)$ *is a weak principal differential calculus on the quantum principal bundle* $\mathcal{F}_G$. *In particular,*

$$0 \to A_I \otimes_{B_I} \Gamma_{B_I} \to \Gamma_{A_I} \xrightarrow{\mathrm{ver}_I} A_I \square_H \Gamma_H \to 0$$

is exact for $I \in \{1, 2, 12\}$.

iv.) *Locally* $(\Upsilon_G, \mathrm{d}_G)$ *is not the smash product calculus, since e.g.*

$$(\Upsilon_G(U_1), \mathrm{d}_1) \ncong (\Gamma_{B_1} \# H \oplus B_1 \# \Gamma_H, \mathrm{d}_\#).$$

Example $\mathrm{GL}_q(2)$ over $\mathbb{CP}^1$

Theorem

The Ore extensions of $A = \mathrm{GL}_q(2)$ *give rise to a quantum principal bundle* $\mathcal{F}$*, namely*

$$\mathcal{F}(\emptyset) = \{0\}, \quad \mathcal{F}(U_1) = A[\alpha^{-1}], \quad \mathcal{F}(U_1) = A[\gamma^{-1}],$$
$$\mathcal{F}(U_1 \cap U_2) = A[\alpha^{-1}, \gamma^{-1}], \quad \mathcal{F}(M) = A.$$

The Ore extension of the bicovariant FODC (Γ_A, d_A) *on* A *is a principal DC* (Υ, d) *on* $\mathcal{F}$*. Locally, it is not the smash product calculus.*

- (Γ_A, d_A) is 4-dimensional $\omega^1, \omega^2, \omega^3, \omega^4$
- The quotient calculus (Γ_H, d_H) on $H = A/\langle\gamma\rangle$ is 3-dimensional $[\omega^1], [\omega^3], [\omega^4]$
- $B_1 = \mathcal{F}(U_1)^{\mathrm{co}H} = \mathbb{C}_q[\alpha^{-1}\gamma]$ with 1-dimensional calculus generated by $\mathrm{d}_1(u) = \mathrm{d}_1(\alpha^{-1}\gamma) = -\alpha^{-2}\omega^2$
- $\mathrm{ver}_1(\sum_{i=1}^4 a^i\omega^i) = \sum_{i=1}^4 a_0^i \otimes a_1^i[\omega^i]$

So $0 \to A_I \otimes_{B_I} \Gamma_{B_I} \to \Gamma_{A_I} \xrightarrow{\mathrm{ver}_I} A_I \Box_H \Gamma_H \to 0$ is exact.

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Thank you for your attention!